1) Complex Numbers

Construction/Defining $\mathbb{C}$ (by Hamilton)

The set of complex numbers denoted $\mathbb{C}$ is the set $\mathbb{R}^2$ together with binary operators,

$$+, \times : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$$

$$\left(\mathbb{R}^2 = \{(x,y) : x, y \in \mathbb{R}^2\}\right)$$

where

$$(a,b) + (c,d) := (a+c, b+d)$$

$$(a,b) \times (c,d) := (ac-bd, ad+bc)$$

So

$$\mathbb{C} = (\mathbb{R}^2, +, \times)$$

When we write $z \in \mathbb{C}$, we understand that

$$z = (a,b) \in \mathbb{R}^2$$

with two binary operations defined above.

We call z a complex number

Suppose $z = (a,b)$ and $w = (x,y)$.

$z = w$ iff $a = x$ and $b = y$

<u>Some Notational devices</u>:

- suppress '$\times$' symbol.
  zw is the same as $z \times w$

- element $(0,1)$ will be written as i.
  So
  $$(0,1) = i$$

- if $z = (a,b) \in \mathbb{C}$ then we say

$$a = \operatorname{Re}(z) \quad \text{and} \quad b = \operatorname{Im}(z)$$

where $a, b \in \mathbb{R}$.

<u>Note</u>:
$$(a,b) = (a,0) + (0,1) \times (b,0)$$
$$= (a,0) + i \times (b,0)$$

Hence <u>denote</u> <u>$(a,b) \in \mathbb{C}$</u> by <u>$a + ib$</u>

- In $a+ib = z \in \mathbb{C}$,

$$a = \mathrm{Re}(z) \qquad b = \mathrm{Im}(z)$$

where $a \in \mathbb{R}$ $\quad b \in \mathbb{R}$

Now routine to derive following expressions:

<u>Addition of complex numbers</u>

Let $z_1 = a+ib$ $\quad z_2 = c+id$

$$z_1 + z_2 := (a+c) + i(b+d)$$

<u>Multiplying complex numbers</u>

Let $z_1 = a+ib$ $\quad z_2 = c+id$

$$\begin{aligned} z_1 \times z_2 &= (a+ib) \times (c+id) \\ &= (a,b) \times (c,d) \\ &= (ac-bd, ad+bc) \\ &= (ac-bd) + i(ad+bc) \end{aligned}$$

The basics of $\mathbb{C}$

Given $z_1, z_2, z_3 \in \mathbb{C}$, the following can be proven:

1) $\exists\, 0 \in \mathbb{C}$ such that $z_1 + 0 = z_1$

   (here $0 = (0,0) \in \mathbb{C}$) $\to$ additive identity

2) $\exists\, -z_1 \in \mathbb{C}$ such that $z_1 + (-z_1) = 0$

   $\hookrightarrow$ additive inverse

3) $z_1 + z_2 = z_2 + z_1$ commutativity

4) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ Associativity

5) $\exists\, 1 \in \mathbb{C}$ such that $z_1 \times 1 = z_1$

   (here $1 = (1,0) \in \mathbb{C}$) $\to$ multiplicative identity

6) $z_1 z_2 = z_2 z_1$ commutativity

7) if $z_1 \neq 0$, then $\exists$ a unique $w \in \mathbb{C}$ such that $z_1 w = 1$

   $\hookrightarrow$ w is the multiplicative inverse

8) $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ distributive laws

Further 0 and 1 as defined before are seen to be

$$0 = 0 + i0$$

$$1 = 1 + i0$$

Subtracting Complex numbers:

let $z_1 = a + ib \quad z_2 = c + id$

$$z_1 - z_2 = (a+ib) - (c+id)$$

$$= (a-c) + i(b-d)$$

Reciprocal of $\mathbb{C}$ (defining $\div$ in $\mathbb{C}$)

Given $z \in \mathbb{C} \setminus \{0\}$ derive formula for w such that $zw = 1$

Let $z = a + ib \neq 0$

$$w = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2} = \frac{1}{a^2+b^2}(a - ib)$$

We want $w = x + iy$ to satisfy:

$$(a+ib)(x+iy) = 1 + 0i$$

$$\Rightarrow (ax - by) + i(ay + bx) = 1 + 0i$$

By equality of complex numbers we get simultaneous eqn:

$$ax - by = 1 \qquad (*1)$$

$$ay + bx = 0 \qquad (*2)$$

Solving:

$$\begin{aligned} ax - by &= 1 \ \times b \Rightarrow & \cancel{ba}x - b^2y &= b \\ ay + bx &= 0 \ \times a & \underset{(-)}{a^2y} + \underset{(-)}{\cancel{ab}x} &= 0 \end{aligned}$$

$$\Rightarrow -b^2y - a^2y = b$$

$$\Rightarrow y = -\frac{b}{a^2+b^2} \rightarrow \text{sub into } (*2)$$

$$a\left(\frac{-b}{a^2+b^2}\right) + bx = 0$$

$$\Rightarrow b\left(x - \frac{a}{a^2+b^2}\right) = 0$$

$$\Rightarrow b = 0 \quad \text{or} \quad x - \frac{a}{a^2+b^2} = 0$$

$$\Rightarrow x - \frac{a}{a^2+b^2} = 0$$

$$\Rightarrow x = \frac{a}{a^2+b^2}.$$

So we found $w = x+iy$ such that $zw = 1$ where

$$w = x+iy = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$$

Using reciprocal of $\mathbb{C}$, we define division.

Dividing complex numbers:

Take $z \in \mathbb{C}$ and $w \in \mathbb{C} \setminus \{0\}$
Let w^{-1} be the unique number s.t $w \cdot w^{-1} = 1$.

Then

$$\frac{z}{w} = z w^{-1}$$

Let $z = a + ib$ $\quad w = c + id \neq 0$

$$\frac{z}{w} = (a+ib)(c+id)^{-1}$$

$$= (a+ib)\frac{(c-id)}{c^2+d^2}$$

$\Rightarrow$

$$\frac{z}{w} = \frac{(a+ib)(c-id)}{c^2+d^2}$$

Lemma: Given $z \in \mathbb{C} \setminus \{0\}$, $z = x + iy$, then

$$\frac{\bar{z}}{|z|^2} = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2} \in \mathbb{C}$$

is the multiplicative inverse of z.

proof: Indeed

$$z \times \frac{\bar{z}}{|z|^2} = \frac{z \times \bar{z}}{|z|^2} = 1$$

and

$$\frac{\bar{z}}{|z|^2} \times z = \frac{\bar{z} \times z}{|z|^2} = 1. \qquad \blacksquare$$

Example: $\frac{z_1}{z_2} = \frac{1+2i}{3-4i}$

$$= \frac{(1+2i)(3+4i)}{|3+4i|^2}$$

$$= \frac{-5+10i}{3^2+4^2} = \frac{-1}{5} + \frac{2}{5} i$$

The modulus and conjugate of complex numbers

Suppose $z = a + ib \in \mathbb{C}$

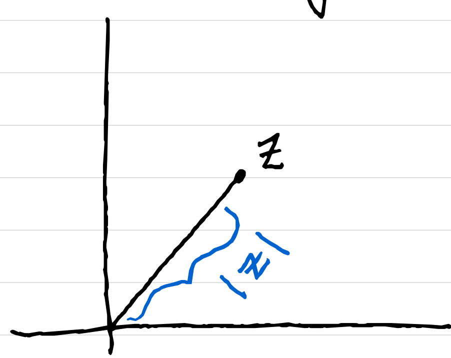
Modulus denoted by $|z|$ of z is

$$\boxed{|z| = \sqrt{a^2+b^2} = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}}$$

Conjugate denoted by $\bar{z}$ of z is

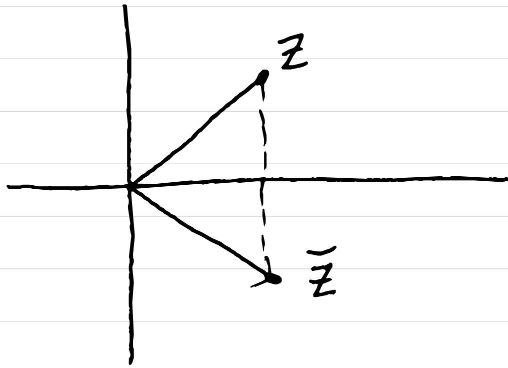
$$\boxed{\bar{z} = a - ib}$$

Geometric meaning of modulus:



Modulus is the distance of z from origin.

Geometric meaning of conjugate:



$\bar{z}$ is the reflection of z on the x-axis

Properties of $\mathbb{C}$:

1) $\boxed{z+\bar{z} = 2\mathrm{Re}(z) \text{ and } z-\bar{z} = 2i\,\mathrm{Im}(z)}$

proof: If $z = x+iy$, $\bar{z} = x-iy$.

$$z+\bar{z} = x+\cancel{iy}+x-\cancel{iy}$$
$$= 2x = 2\mathrm{Re}(z)$$

$$z-\bar{z} = x+iy-(x-iy)$$
$$= \cancel{x}+iy-\cancel{x}+iy$$
$$= 2iy = 2i\,\mathrm{Im}(z)$$

$\boxed{2)\ \overline{z+w} = \overline{z} + \overline{w}}$

proof: Let $z = a+ib$, $w = c+id$, $z, w \in \mathbb{C}$.

$$\overline{z+w} = \overline{(a+ib)+(c+id)}$$
$$= \overline{(a+c)+i(b+d)}$$
$$= a+c - i(b+d)$$
$$= (a-ib) + (c-id)$$
$$= \overline{z} + \overline{w}$$

$\boxed{3)\ \overline{zw} = \overline{z}\,\overline{w}}$

proof: Let $z = a+ib$, $w = c+id$, $z, w \in \mathbb{C}$.

$$\overline{zw} = \overline{(a+ib).(c+id)}$$
$$= \overline{(ac-bd)+i(bc+ad)}$$
$$= (ac-bd) - i(bc+ad)$$
$$= (ac-(-b)(-d)) + i(a(-d)+(b)(-c))$$
$$= (a-ib)(c-id) = \overline{z}.\overline{w}$$

$$\boxed{4)\ \bar{\bar{z}} = z}$$

proof: $\bar{\bar{z}} = \overline{\overline{x+iy}} = \overline{x-iy} = x+iy = z$

$$\boxed{5)\ |z| = |\bar{z}|}$$

proof: Let $z = a+ib$, $z \in \mathbb{C}$. $\bar{z} = x-iy$.

$|z| = \sqrt{x^2+y^2}$

$|\bar{z}|\ \sqrt{x^2+(-y)^2} = \sqrt{x^2+y^2} = |z|$

$$\boxed{6)\ |z|^2 = z\bar{z} = |\bar{z}|^2}$$

proof Let $z = a+ib$, $z \in \mathbb{C}$. $\bar{z} = x-iy$. $a, b \in \mathbb{R}$.

$|z| = \sqrt{a^2+b^2} \Rightarrow |z| = a^2+b^2 \in \mathbb{R}$

$z\bar{z} = (a+ib)(a-ib)$

$= a^2 - \cancel{i}ab + \cancel{i}ab - i^2b^2$

$= a^2 - (-1)b^2 = a^2+b^2 = |z| = |\bar{z}|$

Lemma: If $z \in \mathbb{C} \setminus \{0\}$ and $w \in \mathbb{C}$ is such that

$$zw = 1$$

then

$$w = \frac{1}{|z^2|}\,\overline{z}$$

7) $|zw| = |z| \cdot |w|$

proof: Let $z = a+ib$, $w = c+id$, $z, w \in \mathbb{C}$.

$$|zw|^2 = zw\,\overline{zw}$$

$$= zw\,\overline{z}\,\overline{w}$$

$$= z\overline{z}\,w\overline{w}$$

$$= |z|^2 |w|^2$$

$$= (|z| . |w|)^2$$

$$\Rightarrow |zw| = |z| . |w|$$

$$\boxed{8)\ \text{if } w \neq 0 \text{ then } |z/w| = |z|/|w|}$$

proof: $1/w = \bar{w}/|w|^2$ So

$$\left|\frac{1}{w}\right| = \left|\frac{\bar{w}}{|w|^2}\right| = \frac{|\bar{w}|}{|w|^2} = \frac{|w|}{|w|^2} = \frac{1}{|w|}$$

So $$\left|\frac{z}{w}\right| = |z| \cdot \left|\frac{1}{w}\right| = |z| \cdot \frac{1}{|w|} = \frac{|z|}{|w|}$$

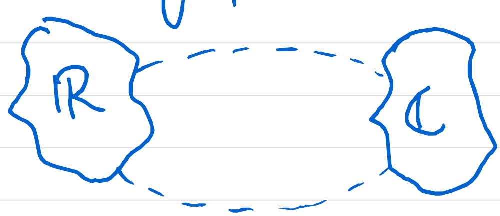
The mysterious i:

Before discussing i, put $\mathbb{R}$ into $\mathbb{C}$.

The fancy nomenclature for this is the canonical embedding of $\mathbb{R}$ into $\mathbb{C}$ denoted by the map

$$\mathbb{R} \hookrightarrow \mathbb{C}$$

↳ basically put $\mathbb{R}$ into $\mathbb{C}$



Done by just this:

$$\psi : \mathbb{R} \to \mathbb{C} : x \mapsto (x, 0)$$

The <u>image set</u> <u>$\psi(\mathbb{R})$</u> is a copy of $\mathbb{R}$ in $\mathbb{C}$.

So

$$\psi(\mathbb{R}) \subseteq \mathbb{C}$$

$\{(x,0) \in \mathbb{C} : x \in \mathbb{R}\}$

So we just accept $\mathbb{R} \subseteq \mathbb{C}$ even if set theoretically it is false. $\psi(\mathbb{R}) \subseteq \mathbb{C}$ is valid.

<u>The mysterious i</u>

Complex numbers are often motivated by a need to solve the quadratic eqn:

$$ax^2 + bx + c = 0$$

so

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To be able to solve this for any real $a \neq 0, b, c$, need to take $\sqrt{\ }$ of negative numbers.

↳ if $b^2 - 4ac < 0$

Simply from defn of $\mathbb{C}$, i is the point $(0,1) \in \mathbb{C}$,

$$i = (0,1) \in \mathbb{C}$$

$$i \times i = (0,1) \times (0,1) = (-1,0) = -1$$

$$\Rightarrow \qquad i^2 = -1$$

Soon becomes clear that we cannot avoid this correspondance idefinately as i is clearly a solution of the polynomial eqn

$$z^2 + 1 = 0$$

$$\hookrightarrow z^2 = -1$$

Define i to be $\sqrt{-1}$

Make it a definition:

$$\boxed{\sqrt{-1} = i}$$

Further for any $b>0$, define $\sqrt{-b}$,

$$\boxed{\sqrt{-b} = i\sqrt{b}}$$

Extend Square root function to $\mathbb{C}$ ($\sqrt{\ }$):

$$\sqrt{\ } : \mathbb{C} \to \mathbb{C} : (x+iy) \mapsto \sqrt{\frac{\sqrt{x^2+y^2}+x}{2}} \pm i\sqrt{\frac{\sqrt{x^2+y^2}-x}{2}}$$

where $\pm$ is chosen to be the sign of y if $y \neq 0$ or $+$ if $y=0$

Caution of $\sqrt{\ }$ in $\mathbb{C}$

Care has to be taken when manipulating this extended square root function.

For example the following properties are **NOT** true on whole of $\mathbb{C}$

1) $\sqrt{zw} = \sqrt{z}\sqrt{w}$
2) $\sqrt{z/w} = \sqrt{z}/\sqrt{w}$

True in $\mathbb{R}$
not true in $\mathbb{C}$

This is due to discontinuity in function when extended to complex plane. [See functions of complex variable].

Ordering on $\mathbb{C}$

No analogue of $\leq$ (or $<$) in $\mathbb{C}$.

$\mathbb{C}$ has no total ordering in $\mathbb{C}$ which is compatible with $+$ and $\times$

Reason for no order:

Q: is $i > 0$. Assume yes. $\Rightarrow i \times i > 0$
$\Rightarrow -1 > 0$, inconsistant

$i < 0$ Assume yes $\Rightarrow -i \times -i > 0 \Rightarrow -1 > 0$
inconsistent.

Statements like:

"Let $z, w \in \mathbb{C}$ with $z \leq w$" are FALSE

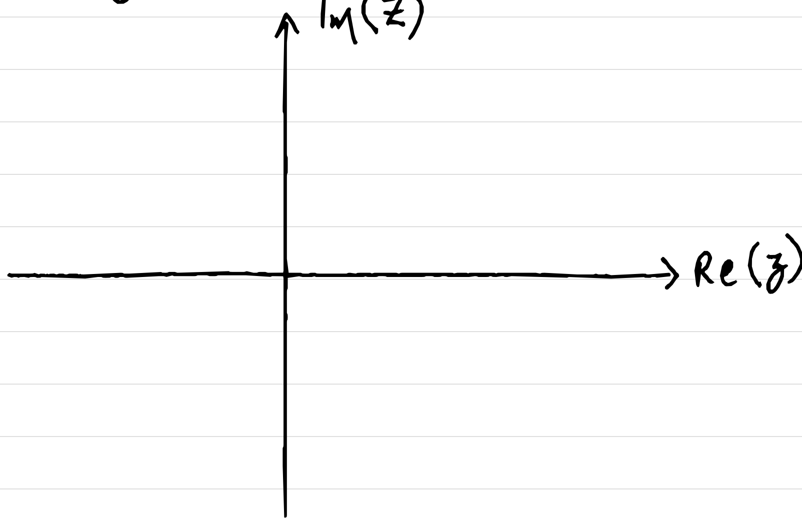
OK though to use $\leq$ (or $<$) when comparing real values such as $|z|$, $|w|$, $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, $\mathrm{Re}(w)$, $\mathrm{Im}(w)$ etc.

$\mathrm{Re}(z)$ and $\mathrm{Im}(z) \in \mathbb{R}$.

The Geometry of $\mathbb{C}$:

The Argand Plane: (fancy name for $\mathbb{R}^2$)

The Argand plane is the "complex plane



In the argand plane,

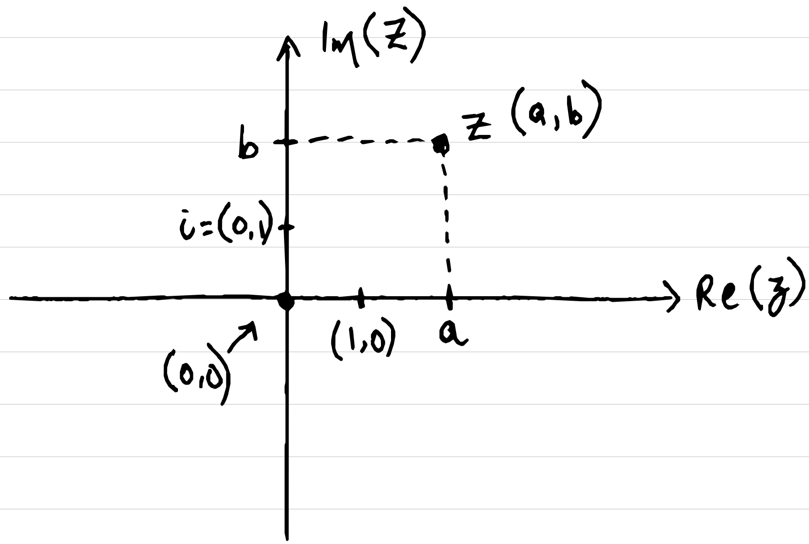
- x - axis is the real axis
- y - axis is the imaginary axis

Argand plane relies on the correspondance between $z = a + ib$ and the point (a, b) as a real 2-ple.

Since $z = a + ib = Re(z) + Im(z)$

$$= (\underset{a}{\underset{\parallel}{Re(z)}}, \underset{b}{\underset{\parallel}{Im(z)}}),$$

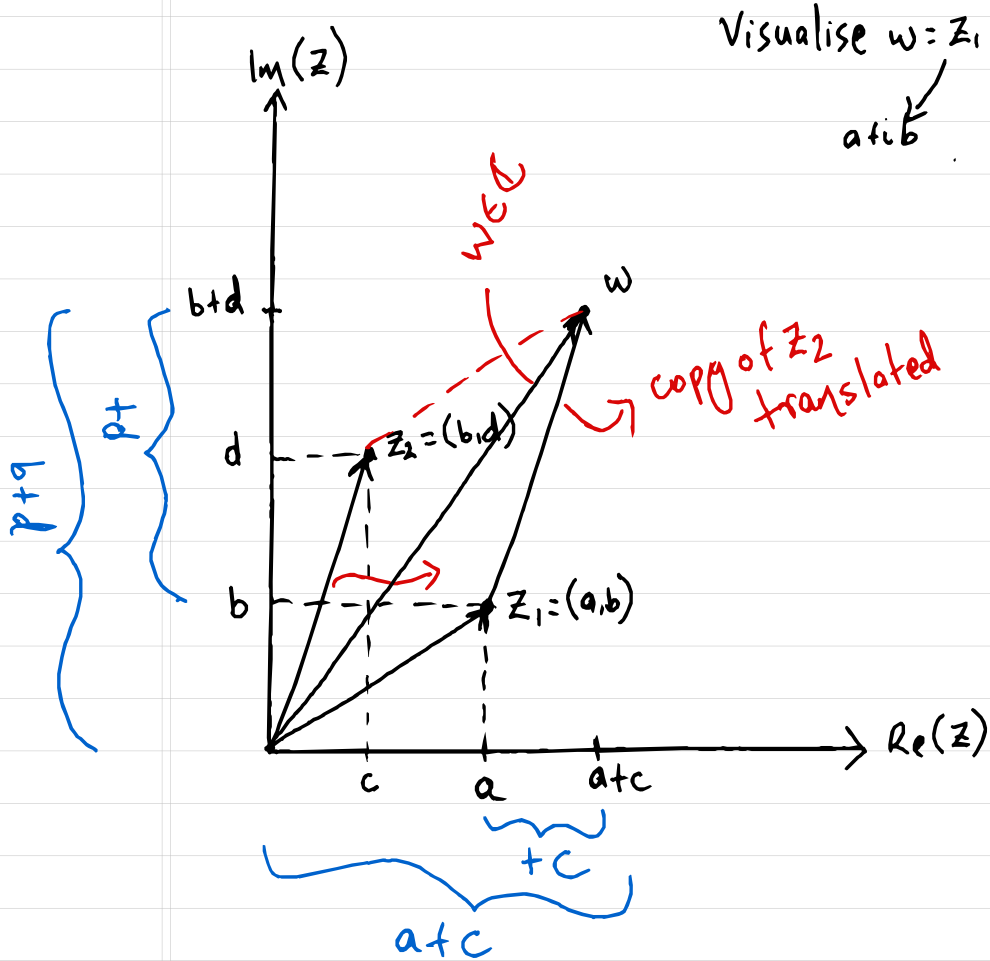
plot $z = a + ib$ as co-ordinates.



<u>Geometrical interpretation of addition</u>:

Visualise $w = z_1 + z_2$

$z_1 \to a + ib$, $z_2 \to c + id$



If the parallelogram by the two dimensional vectors which represent the complex numbes The addition of them is what happens when you translate z_2 onto z_1 and add z_2 as a vector to z_1.

Here w is the <u>diagonal of parallelogram formed by z_2 and z_1</u>.

Example: Calculate length of diagonal of parallelogram with vertices at the origin $(1,1)$, $(-2,3)$.

Ans: The desired length is $|w|$ where

$$w = (1+i) + (-2+3i) = 1+4i$$

(As adding two complex numbers is the diagonal of parallelogram)

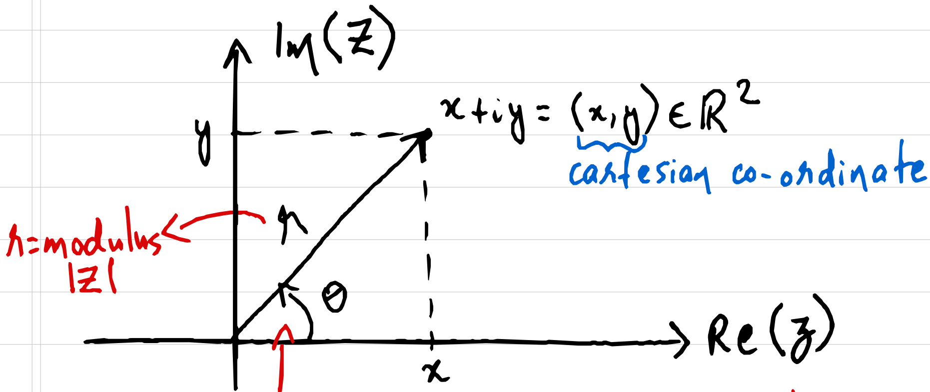
So length of diagonal is

$$|w| = |-1+4i|$$

$$= \sqrt{1^2+4^2}$$

$$= \sqrt{17}$$

Polar form of Complex Numbers:



argument = θ measured anticlockwise
anticlockwise = positive
clockwise = negative.

$\boxed{\theta \text{ always measured in radians}}$

$r \in [0, \infty) \Rightarrow r$ is a strictly non-negative number.

$\theta \in \mathbb{R}$

- $\underline{\theta > 0} \Rightarrow$ angle is measured <u>anti-clockwise</u>
- $\underline{\theta < 0} \Rightarrow$ angle is measured <u>clockwise</u>

In polar, we will write $z \in \mathbb{R}^2$ as (r, θ)

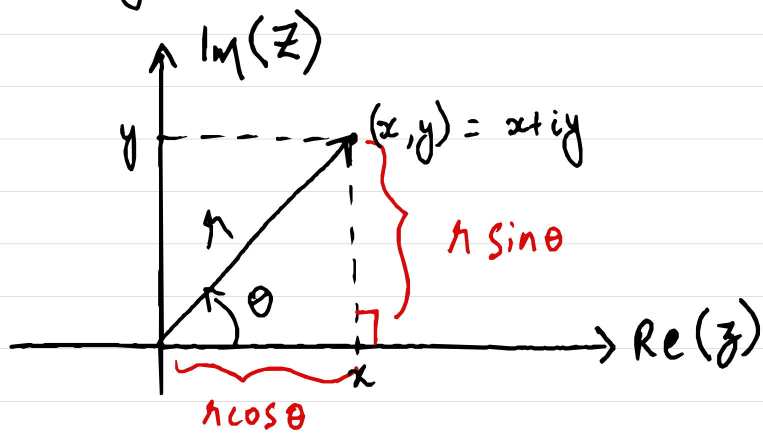
↳ Alternative notation
$r \angle \theta$

Cartesian co-ordinates <u>are unique.</u>

Polar co-ordinates <u>are not unique</u>

$$(r, \theta) = (r, \theta + 2k\pi) \quad \text{for } k \in \mathbb{Z}$$

<u>Converting from polar to cartesian:</u>



given r and θ; (derived using trigonometry)

$$\boxed{x = r\cos\theta \qquad y = r\sin\theta}$$

Note:

$$x^2 + y^2 = r^2 \Rightarrow r = \sqrt{x^2 + y^2} = |z|$$

Remember that our value of $\underline{\underline{\theta}}$ is <u>not</u> <u>unique</u>.

$\cos\theta = \cos(\theta + 2k\pi)$, $\sin\theta = \sin(\theta + 2k\pi)$ for some $k \in \mathbb{Z}$

So we <u>restrict our θ</u> and this leads to the <u>principle argument</u>

Convention, take

$$\theta \in (-\pi, \pi]$$

and call it the <u>principle argument</u>.

$0 \leq \theta \leq \pi \Rightarrow$ anti-clockwise

$-\pi < \theta \leq 0 \Rightarrow$ clockwise.

So

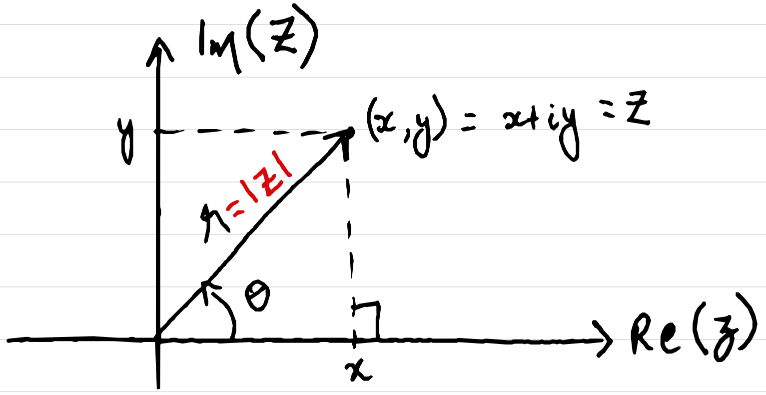
$$z = x + iy$$

$$= r\cos\theta + ir\sin\theta$$

$$= r(\cos\theta + i\sin\theta)$$

Converting from cartesian to polar:

From (x, y) to (r, θ).



We know $\boxed{r = \sqrt{x^2 + y^2} = |z|}$

From trigonometry, it is clear that

$$\tan\theta = \frac{y}{x}, \quad x \neq 0.$$

But the problem is

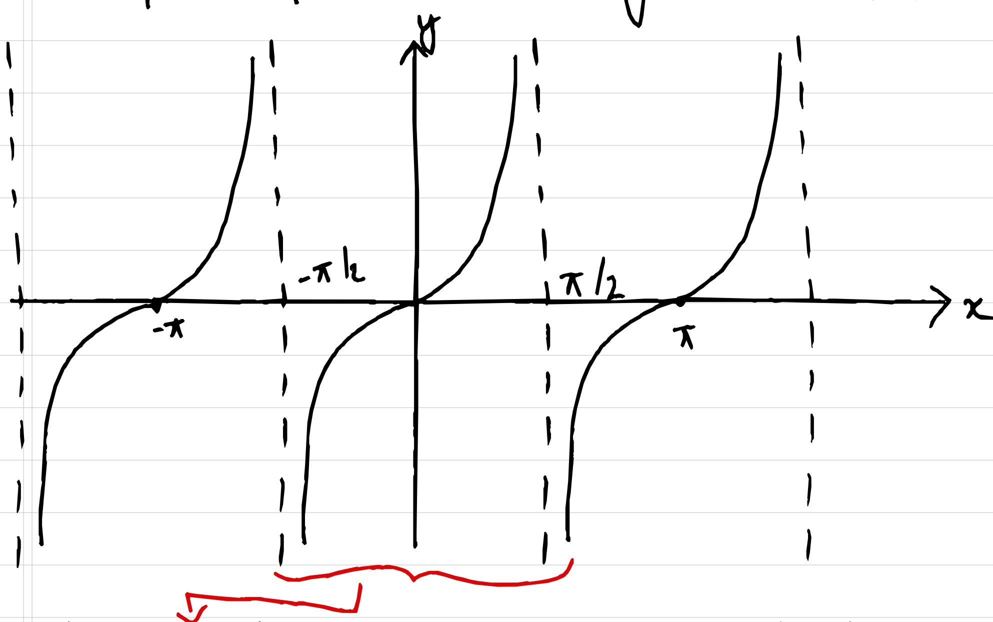
$\tan\theta = \frac{y}{x}$, $x \neq 0$ does not always imply

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

i.e.

$$\tan\theta = \frac{y}{x} \not\Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

The problem is due to way tan is defined.



When we define tan inverse, we restrict our domain to $-\pi/2, \pi/2$

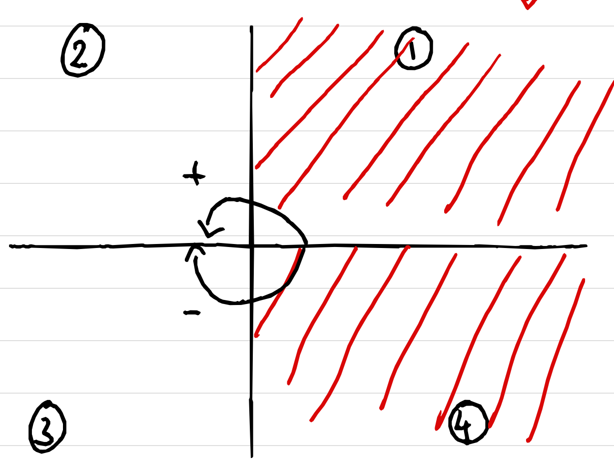
Reason to restrict domain:
In order to define an inverse functions such as $\tan^{-1}$, the function (such as tan) <u>needs to become one-to-one and onto</u>.

$\tan : \mathbb{R} \to \mathbb{R}$ is <u>not one-to-one</u> but

$\tan : [-\pi/2, \pi/2] \to \mathbb{R}$ is <u>one to one and onto</u>

So the inverse tan would be:

$$\boxed{\tan^{-1} : \mathbb{R} \to [-\pi/2, \pi/2]}$$



This causes the output of the $\tan^{-1}$ function to be restricted to quadrants 1 and 4

$\boxed{\text{So if we just apply } \tan^{-1}(y/x) \text{ directly, we would not get argument values for quadrants 2 and 3}}$

So we need to draw graphs and use geometry.

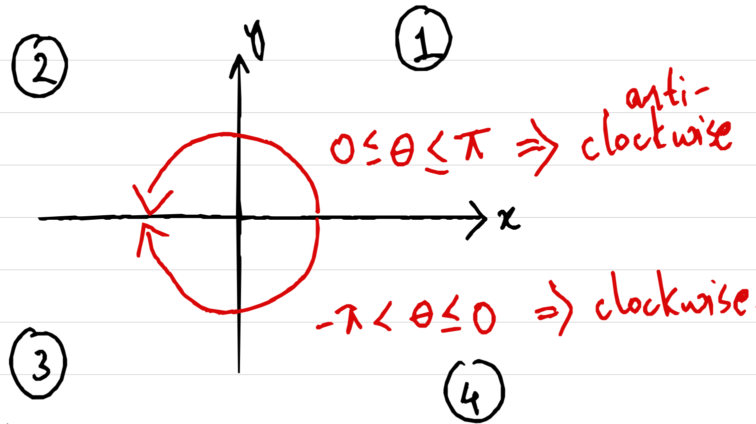
Notation:
principle argument values are represented by capital a (A)

$$Arg(z)$$

other argument values by small a

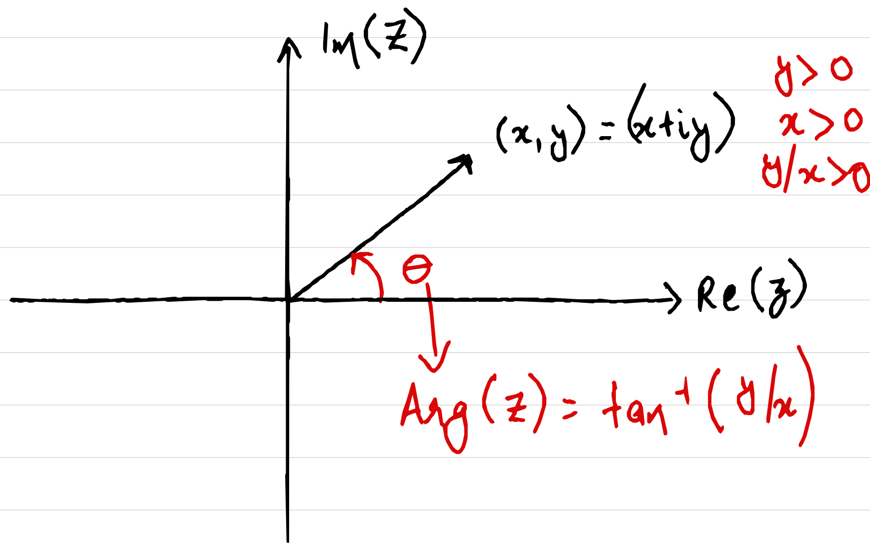
$$arg(z)$$

Convention: $Arg(z) \in (-\pi, \pi] = \{\theta \in \mathbb{R} \mid -\pi < \theta \leq \pi\}$



Calculating Arguments:

- if $y>0$, $x>0$, $y/x>0$, $\tan^{-1}(y/x)$ would give answer in quadrant 1

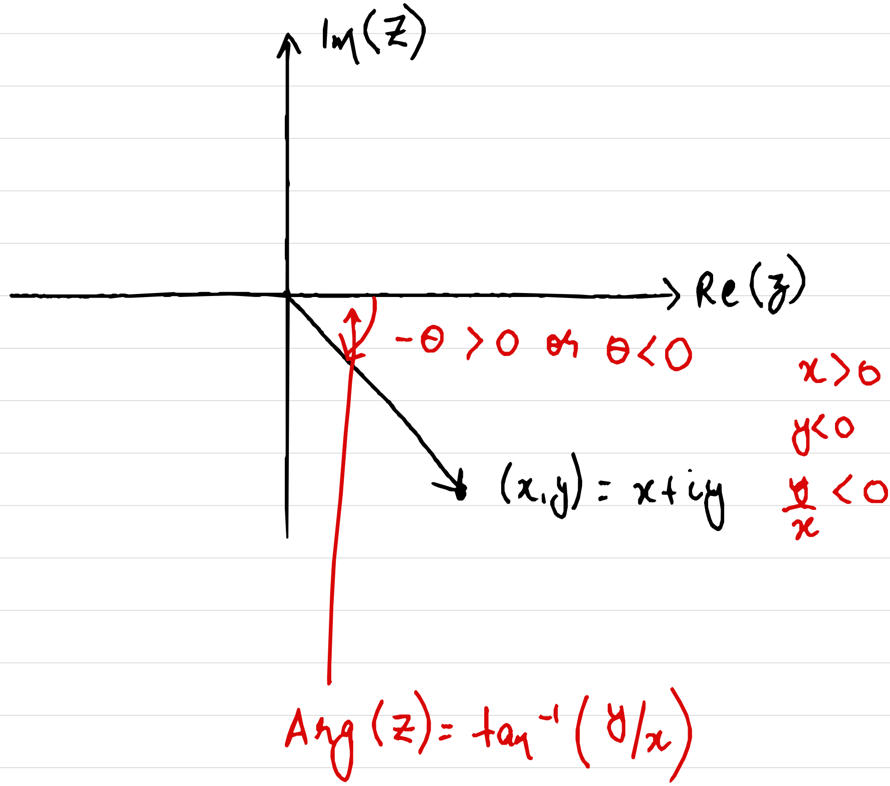


So here,

$$Arg(z)=\tan^{-1}(y/x)$$

- if $\underline{x>0 \text{ and } y\leq 0}$, $\underline{y/x<0}$, $\tan^{-1}(y/x)$ would give θ in quadrant 4 ✓

$(\tan(-\theta) = -\tan\theta)$ Here θ



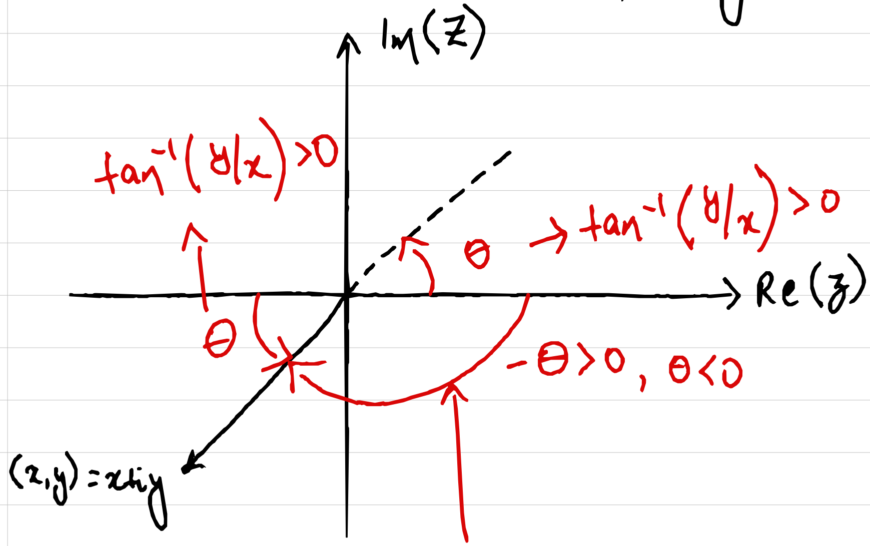
So here

$$Arg(z) = \tan^{-1}(y/x)$$

- if $\underline{y<0}$, $\underline{x<0}$, then $\underline{y/x>0}$, $\tan^{-1}(y/x)$ would give answer in first quadrant due to $\tan^{-1}$ defn.

But we need argument value in principle argument pertaining to third quadrant 3.

So we add $-\pi$ to obtain principle argument.



$x<0$

$y<0$

$\frac{y}{x}<0$

$$Arg(z) = -\pi + \theta$$

$$= -\pi + \tan^{-1}(y/x)$$

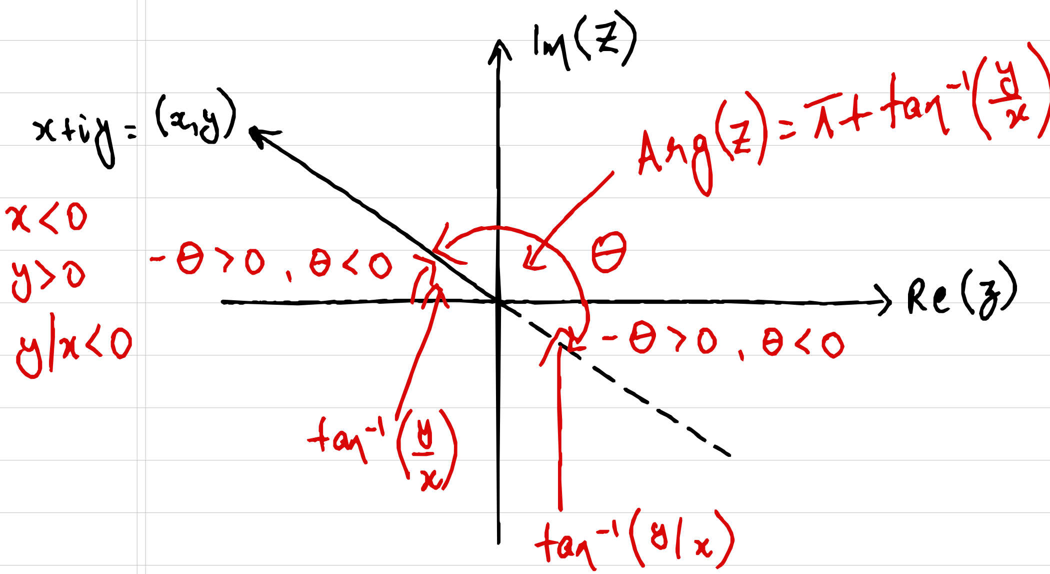
So here

$$Arg(z) = -\pi + \tan^{-1}\left(\frac{y}{x}\right)$$

- if $y>0$, $x<0$, then $y/x<0$, $\tan^{-1}(y/x)$ would give answer in fourth quadrant due to $\tan^{-1}$ defn.

But we need argument value in principle argument pertaining to third quadrant (2)

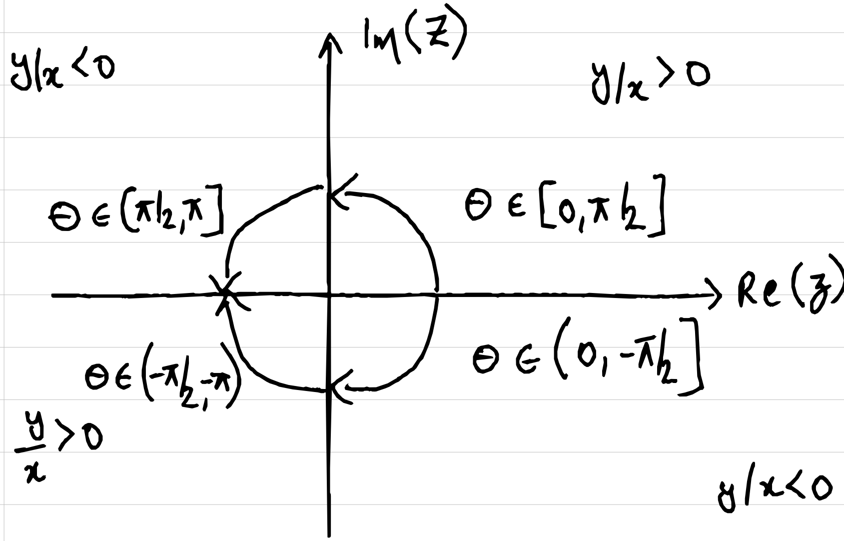
So we add π to obtain principle argument.

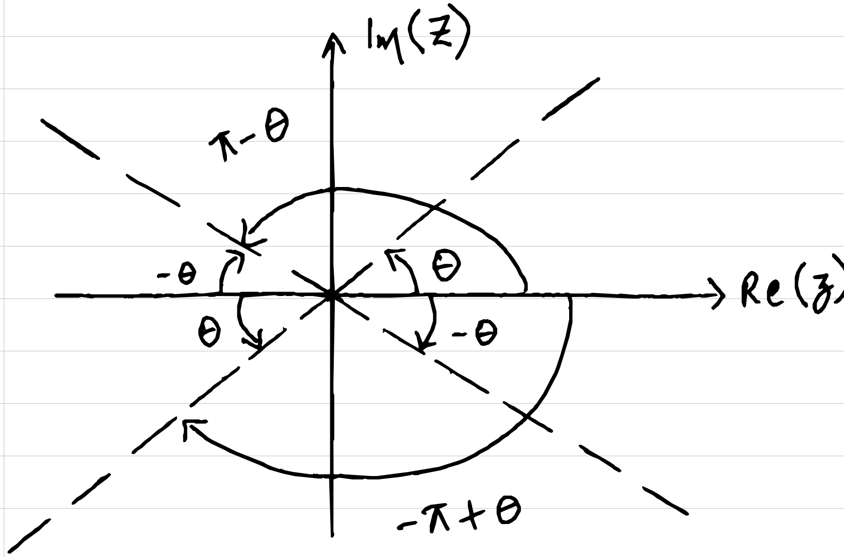


So here

$$\mathrm{Arg}(z) = \pi + \tan^{-1}\left(\frac{y}{x}\right)$$

Summary of Argument Calculation:





The geometry of complex multiplication:

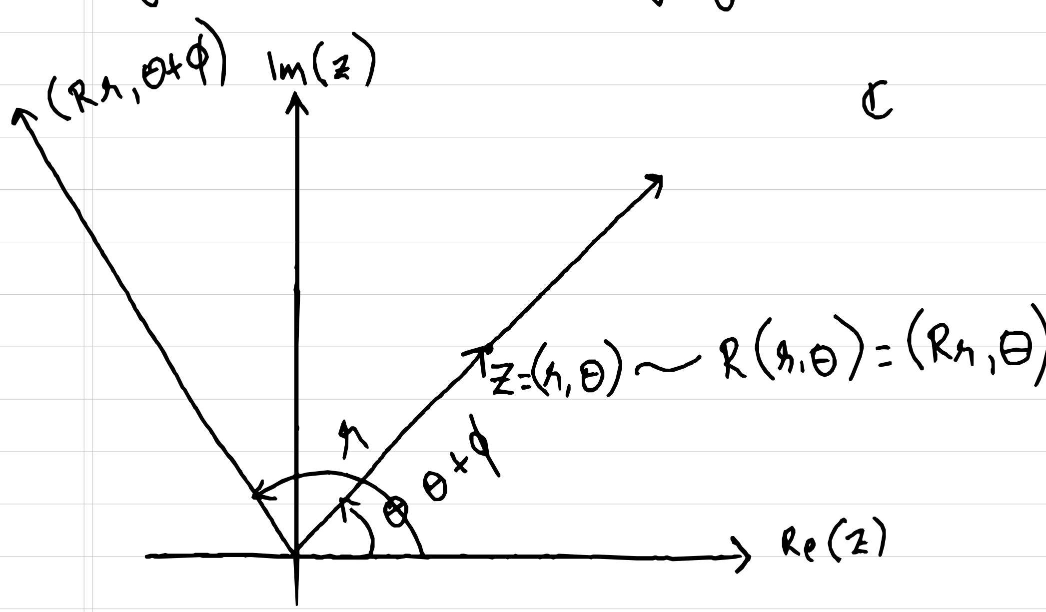
In terms of geometry, $\times$ operation on $\mathbb{C}$ is rotation and scaling.

Fix a positive real number $R > 0$ and an angle $\phi \in (-\pi, \pi]$

Then multiplication by the complex number

$$w = R(\cos\phi + i\sin\phi)$$

corresponds to rotating the complex plane through angle ϕ and then scaling by factor R.



Take $z = r\cos\theta + ir\sin\theta$ $\qquad z = (r, \theta)$

$w = R\cos\phi + iR\sin\phi$ $\qquad w = (R, \phi)$

$$zw = (r\cos\theta + ir\sin\theta)(R\cos\phi + iR\sin\phi)$$

$$= rR\cos\theta\cos\phi + rRi\cos\theta\sin\phi + irR\sin\theta\cos\phi - rR\sin\theta\sin\phi$$

$$= rR(\cos\theta\cos\phi - \sin\theta\sin\phi) +$$
$$irR(\cos\theta\sin\phi + \sin\theta\cos\phi)$$

Using double angle formulae

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

$$= rR\cos(\theta + \phi) + irR\sin(\theta + \phi)$$

$$= rR(\cos(\theta + \phi) + i\sin(\theta + \phi))$$

So

$$\boxed{zw = rR(\cos(\theta+\phi) + i\sin(\theta+\phi))}$$

$$zw = rR(\cos(\theta+\phi) + i\sin(\theta+\phi))$$

multiply moduli
<u>contraction or</u>
<u>expansion</u>

add arguments
<u>rotation</u>

So if we have complex number multiplication,

$$z = (r, \theta), \quad w = (R, \phi)$$

$$zw = (rR, \theta+\phi)$$

Basically <u>multiply moduli</u> : contraction or expansion

add arguments : rotation.

Euler Form of Complex Numbers:

Euler (1740) is credited with the following important relationship.

Let e be the usual for natural logarithms, the

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

The complex number $z = r(\cos\theta + i\sin\theta)$ can be written more compactly as

$$\boxed{z = r(\cos\theta + i\sin\theta) = re^{i\theta}}$$

A.K.A the polar form of z.

Note:

$$|z| = r$$

$$|e^{i\theta}| = 1$$

$$|z|^2 = r^2\cos^2\theta + r^2\sin^2\theta$$
$$= r^2(1) \Rightarrow |z|^2 = r^2$$
$$\Rightarrow |z| = r \quad (r > 0)$$

Multiplication with Euler form and Euler Formula:

Rule of thumb:
Treat $e^{i\theta}$ like you would treat e^x where $x \in \mathbb{R}$

As above, if $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ and
$w = R(\cos\phi + i\sin\phi) = Re^{i\phi}$ then

$$\boxed{zw = rRe^{i(\theta+\phi)}}$$

with a possible adjustment that

$$\theta + \phi \in (-\pi, \pi]$$

add arguments.

$$zw = rRe^{i(\theta+\phi)}$$

multiply moduli

Possible that $\theta + \phi > 2\pi$. In that case

$$e^{i(\theta+\phi)} = e^{i(\theta+\phi-2\pi)}$$

Note:

$e^{i\theta}$ is 2π periodic.

$$e^{i\theta} = e^{i(\theta + 2k\pi)} \quad \text{for some } k \in \mathbb{Z}$$

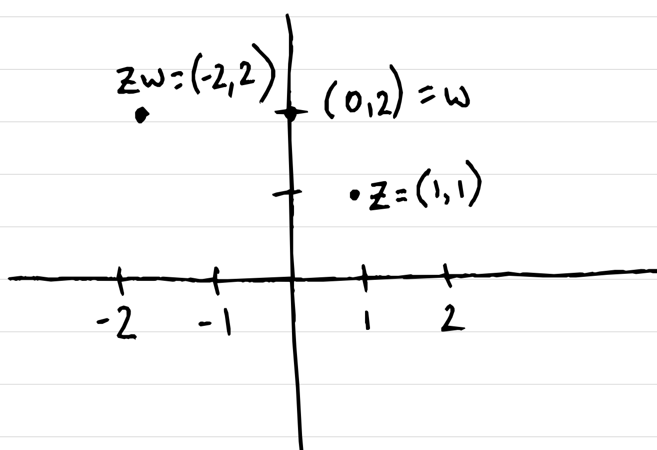
Example: Multiply $z = 1 + i$ and $w = 2i$

Arithmetically:

$$zw = (1+i)(2i)$$

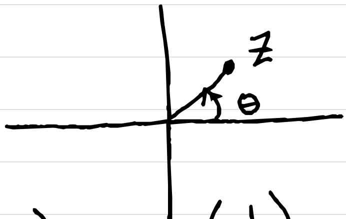
$$= 2i + 2i^2$$

$$= -2 + 2i = (-2, 2)$$



Polar co-ordinates:

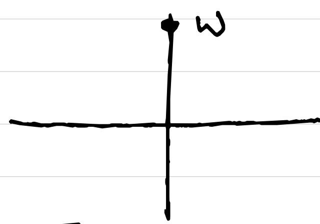
$z = 1 + i$, $r = \sqrt{1^2 + 1^2} = \sqrt{2}$



$\text{Arg}(z) = \tan^{-1}\left(\frac{1}{1}\right) = \pi/4$

$z = \sqrt{2}\, e^{i\pi/4}$

$w = 2i$



$r = \sqrt{2^2} = 2$

$\text{Arg}(z) = \pi/2$

$w = 2e^{i\pi/2}$

So $z = \sqrt{2} e^{i\pi/4}$ $\quad w = 2e^{i\pi/2}$

$$(zw) = 2\sqrt{2}\, e^{i(\pi/4 + \pi/2)}$$

$$= 2\sqrt{2}\, e^{i(3\pi/4)}$$

Verifying answer:

$$zw = 2\sqrt{2}\, e^{i(3\pi/4)}$$

$$= 2\sqrt{2}\left(\cos(3\pi/4) + i\sin(3\pi/4)\right)$$

$$= 2\sqrt{2}\left(\frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

$$= -2 + 2i$$

which is consistent with arithmetic method.

Simplifying statements using Polar form.

If $z = re^{i\theta}$ then

$$\boxed{\bar{z} = re^{-i\theta}}$$

and

$$\boxed{\frac{1}{z} = \frac{1}{r}e^{-i\theta}} \quad \text{if } r \neq 0$$

Lemma: We have:

$$\boxed{\begin{aligned} \cos(x) &= \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \\ \sin(x) &= \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \end{aligned}}$$

proof: Recall that $e^{ix} = \cos(x) + i\sin(x)$, so if $x \in \mathbb{R}$,

$$\cos(x) = \mathrm{Re}(e^{ix}) \quad \text{and} \quad \sin(x) = \mathrm{Im}(e^{ix})$$

Also recall that $\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z})$, $\mathrm{Im}(z) = \frac{1}{2}(z - \bar{z})$

Required result follows from fact $\overline{e^{i\theta}} = e^{-i\theta}$ ∎

Heuristic Argument to why Euler formula holds:

Using power series:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x + i\sin x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$+ i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\right)$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \dots$$

$(*_1)$

Remember that:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Replace x with ix and use power series on e^{ix}

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix^4)}{4!} + \frac{(ix)^5}{5!} + \cdots$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} + \cdots$$

(*2)

So it seems that by (*1) and (*2)

$$e^{ix} = \cos x + i \sin x$$

Extending exponential to $\mathbb{C}$:

Suppose $z = x + iy$. Then

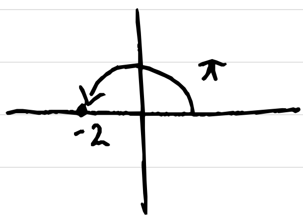
$$\boxed{e^z = e^{x+iy} = e^x \cdot e^{iy}}$$

Example: Solve $e^z = -2$.

Ans: We need to find all pairs $(x, y) \in \mathbb{R}^2$ s.t

$$e^z = e^x \cdot e^{iy} = -2.$$

-2 in polar form:



$r = \sqrt{(-2)^2} = 2 \quad \theta = \pi$

$$-2 + 0i = 2e^{i\pi} = 2e^{i(\pi + 2k\pi)} \quad \text{for } k \in \mathbb{Z}$$

So

$$e^x \cdot e^{iy} = 2e^{i(\pi + 2k\pi)}$$

$$\Rightarrow \quad e^x = 2 \quad \text{and} \quad y = \pi + 2k\pi \quad \text{for } k \in \mathbb{Z}$$

$e^x = 2 \Rightarrow x = \ln 2$

Therefore $x = \ln 2$ $y = \pi + 2k\pi$ for $k \in \mathbb{Z}$

So

$$z_k = \ln 2 + i(\pi + 2k\pi) \quad , \quad k \in \mathbb{Z}$$

So $z_0 \neq z_2 \neq z_3 \neq \cdots \neq z_k$

$\hookrightarrow$ as imaginary parts are different.

$\operatorname{Im}(z_k) = \operatorname{Im}(z_l) \Longleftrightarrow k = l$.

<u>Extending sin and cos to complex numbers</u>:

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

Here $z = x + iy \in \mathbb{C}$.

De Moivre's Theorem:

The following result is attributed to de Moivre:

If $n \in \mathbb{N}$ then

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)}$$

or more compactly:

$$\boxed{(e^{i\theta})^n = e^{in\theta}}$$

proof: (by principle of induction):

Base case $n=1$:
The result follows immediately as

$$\cos\theta + i\sin\theta = \cos(1.\theta) + i\sin(1.\theta)$$

Assume it is true for $n = k \in \mathbb{N}$, inductive hypothesis, i.e.

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

Showing that:

if property is true for $n = k \in \mathbb{N}$, then it is true for $n = k+1 \in \mathbb{N}$

(inductive step):

$$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^{k}(\cos\theta + i\sin\theta)$$

$$= (\cos(k\theta) + i\sin(k\theta))(\cos\theta + i\sin\theta)$$

(by inductive hypothesis)

$$= \cos(k\theta)\cos\theta + i\sin(k\theta)\cos\theta + i\sin\theta\cos(k\theta) - \sin(k\theta)\sin\theta$$

$$= (\cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta) + i(\sin(k\theta)\cos\theta + \sin\theta\cos(k\theta))$$

$$= \cos(k\theta + \theta) + i\sin(k\theta + \theta)$$

$$= \cos(\theta(k+1)) + i\sin(\theta(k+1))$$

$$\Rightarrow (\cos\theta + i\sin\theta)^{k+1} = \cos((k+1)\theta) + i\sin((k+1)(\theta))$$

$\blacksquare$

An application of de Moivre:

Find a formula for $\cos 3\theta$ in $\cos\theta$

By de Moivre, we know

$$(\cos\theta + i\sin\theta)^3 = \cos(3\theta) + i\sin(3\theta)$$

real part   imaginary part

Thus

$$\mathrm{Re}\left((\cos\theta + i\sin\theta)^3\right) = \cos 3\theta$$

(Two complex numbers are same if real and imaginary parts are equal).

$$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + \binom{3}{1}\cos^2\theta(i\sin\theta)^1$$

(binomial theorem) $+\binom{3}{2}\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$

$$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$

$$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

It follows that

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

$$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$$

$$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$$

$$\Rightarrow \boxed{\cos 3\theta = 4\cos^3\theta - 3\cos\theta}$$

Roots of Unity

Let $n \in \mathbb{N}$. A complex number ζ is an n^{th} root of unity if

$$z^n = 1$$

de Moivre $\Rightarrow$ $\zeta = e^{i\left(2\pi \frac{j}{n}\right)}$ where $j = 0, 1, 2, \ldots, n-1$, are solutions of the equation.

$\rightarrow \theta$ not restricted to principle argument

why?
Let
$$\zeta = e^{i(2\pi j/n)} \Rightarrow \zeta^n = \left(e^{i\frac{2\pi j}{n}}\right)^n$$

$$\Rightarrow \zeta = e^{i(2\pi j)} \qquad j \in \mathbb{Z}$$

$$\Rightarrow \zeta = 1$$

$e^{i(0+2\pi j)}$

Are there any more other than $j = 0, 1, \ldots, n-1$?
No, well not without repeat of roots.

$\zeta = e^{i(2\pi \frac{j}{n})}$ where $j = 0, 1, \cdots, n-1$

$\Rightarrow \theta = 0, 2\pi \cdot \frac{1}{n}, 2\pi \cdot \frac{2}{n}, 2\pi \cdot \frac{3}{n}, \cdots, 2\pi \cdot \frac{n-1}{n} \in [0, 2\pi)$

Lemma: $\zeta_j := e^{i(2\pi \frac{j}{n})}$:

for any $0 \leq j < j' \leq n-1$ $\zeta_j \neq \zeta_{j'}$

meaning all roots are distinct.

proof (by contradiction) Since $j' \neq j$, and $j < j'$, then for some $k \in \mathbb{Z} \setminus \{0\}$

$$j' = j + k.$$

Suppose $\zeta_j = \zeta_{j'}$

$$\Rightarrow e^{i(2\pi j/n)} = e^{i(2\pi j'/n)}$$

$$\Rightarrow e^{i(2\pi j/n)} = e^{i(2\pi (j+k)/n)}$$

$$\Rightarrow \cancel{e^{i(2\pi j/n)}} = \cancel{e^{i(2\pi j/n)}} \cdot e^{i(2\pi k/n)}$$

$$\Rightarrow e^{i(2\pi k/n)} = 1$$

This is only possible if k is a multiple of n. its the only possibility.

But k can't be a multiple of n.
It cannot be zero otherwise $j = j'$ which goes against hypothesis.

It can't be n otherwise $j+k > n$ but according to hypothesis,

$$j + k = j' \leq n-1.$$

So we have a contradiction.

↯

So $\zeta_j \neq \zeta_{j'}$

$\blacksquare$

Geometrical interpretation: vertices of a regular n-gon with vertex at $(1,0)$ and all rest on a circle with radius 1 centre the origin.

Explanation of geometry:

$\zeta_0 = 1 = e^{i0}$

+ angle $2\pi/n$

$\zeta_1 = e^{i\frac{2\pi}{n}}$

+ angle $2\pi/n$

$\zeta_2 = e^{i\left(\frac{2\pi}{n} + \frac{2\pi}{n}\right)} = e^{i\left(\frac{2\pi}{n}.2\right)}$

+ angle $2\pi/n$

$\zeta_3 = e^{i\left(\frac{2\pi}{n}.2 + \frac{2\pi}{n}\right)} = e^{i\left(\frac{2\pi}{n}.3\right)}$

+ angle $2\pi/n$

$\zeta_4 = e^{i\left(\frac{2\pi}{n}.3 + \frac{2\pi}{n}\right)} = e^{i\left(\frac{2\pi}{n}.4\right)}$

+ angle $2\pi/n$

$\vdots$

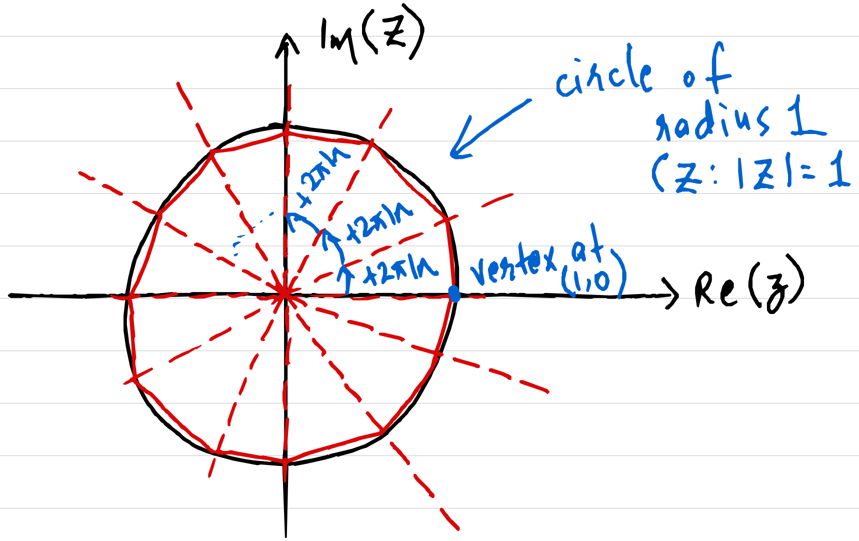
+ angle $2\pi/n$

$\zeta_{n-1} = e^{i\left(\frac{2\pi}{n}.n-2 + \frac{2\pi}{n}\right)} = e^{i\left(\frac{2\pi}{n}.(n-1)\right)}$

Note that

$|\zeta_j| = 1.$

So on a diagram, it looks like



As you can see it forms a regular n-gon inside a unit circle.

Example: Calculate the third roots of unity

$$z^3 = 1 \quad (\text{or } z^3 - 1 = 0)$$

Algebraic method:

$$(z^3 - 1) = (z-1)(z^2 + z + 1)$$

finds roots of $z^2 + z + 1$ to calculate other solutions using quadratic formula

Using roots of unity method:

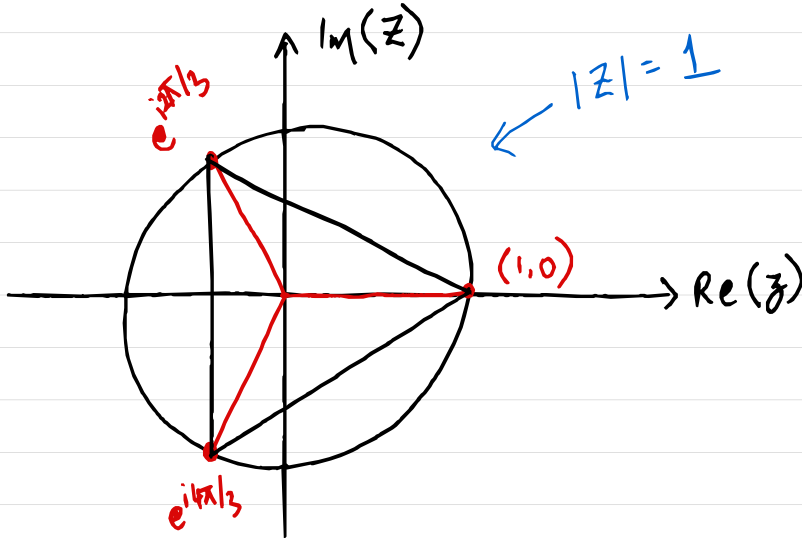
$$z^3 = 1 \Rightarrow \zeta_j = 2^{i(2\pi j/3)} \quad j = 0, 1, 2$$

$$\Rightarrow \zeta_0 = 1$$

$$\zeta_1 = e^{2\pi/3} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$\zeta_2 = e^{4\pi/3} = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Draw on graph next page



regular 3-gon ⇒ triangle

Lemma: for any $n>2$, n^{th} roots of unity sum to 0, in the sense that

$$\sum_{j=1}^{n} \zeta_j^{(n)} = 0$$

and they might multiply to give

- either -1 if n is even
- either $+1$ if n is odd in the sense that

$$\prod_{j=1}^{n} \zeta_j^{(n)} = (-1)^{n+1}$$

Example: for cube roots of unity, found before,

$$1+\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)+\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)=0$$

and

$$1\cdot\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)\cdot\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)$$

$$=\frac{1}{4}+\frac{3}{4}=1.$$

proof: The roots of unity are by definition, the solutions to the polynomial equation:

$$z^n-1=0.$$

Therefore we can write

$$(z^n-1)=(z-\zeta_1^{(n)})(z-\zeta_2^{(n)})\dots(z-\zeta_n^{(n)})$$

and expanding out the brackets, and finding the co-efficients of z^{n-1} and z^0, we see that:

(next page)

- The co-efficients of the z^{n-1} term on the RHS is the sum of roots of unity, and since for $n>1$, there is no term z^{n-1} so

$$\text{RHS coefficient} = \sum_{j=1}^{n} \zeta_j^{(n)} = 0 = 0 \cdot z^{n-1}$$

$$= \text{LHS coefficient.}$$

- The coefficient of constant term on RHS is the product

$$\prod_{j=1}^{n} \left(-\zeta_j^{(n)}\right)$$

and constant term in defining eq $= -1$

So
$$\prod_{j=1}^{n} \left(-\zeta_j^{(n)}\right) = (-1)^n \prod_{j=1}^{n} \zeta_j^{(n)}$$

Hence we have

$$-1 = (-1)^n \left(\prod_{j=1}^{n} \zeta_j^{(n)}\right)$$ and so we have

$$(-1)^{n+1} = \prod_{j=1}^{n} \zeta_j^{(n)} \qquad \blacksquare$$

Complex roots in general

The method for roots of unity works just as well for

$$\boxed{z^n = R \quad \text{where } R \in \mathbb{R} > 0}$$

Let $n \in \mathbb{N}$. the n^{th} roots $R > 0$ are the n distinct complex numbers

$$\zeta_j = R^{1/n} e^{i\left(2\pi \frac{j}{n}\right)}$$

where $j = 0, 1, 2, \ldots, n-1$.

This is just basically from de Moivre again.

Suppose $z = re^{i\theta}$ and $z^n = R$ then by de Moivre,

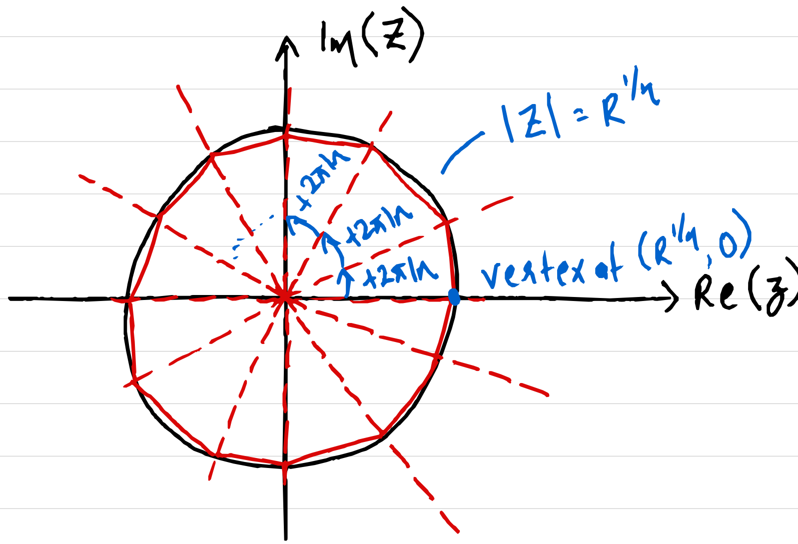
$$r^n e^{in\theta} = Re^{i2\pi k}, \quad k \in \mathbb{Z}$$

So $r = R^{1/n}$ & $\theta = 2\pi \frac{k}{n}$, $k = 0, 1, \ldots, n-1$

for distinct values.

Note the repeats when $k = \lambda n$.

Geometry of roots of $z^n = R$ = vertices of a polygon, an n gon with one vertex at $(R^{1/n}, 0)$



We can also find $z^n = c$ where $c \in \mathbb{C}$

Roots of $z^n = c$ where $c \in \mathbb{C}$:

Asked to find the n^{th} roots of $c \in \mathbb{C}$: find solutions to the equation

$$z^n = c \Rightarrow (z^n - c) = 0$$

Expressing c in polar form:

Let $c = re^{i\arg(c)}$. Then the n^{th} roots of $c \in \mathbb{C}$ are the n distinct complex numbers

$$\zeta_j = r^{1/n} e^{i\left(\frac{\arg(c)}{n} + 2\pi\frac{j}{n}\right)}$$

where $j = 0, 1, 2, \ldots, n-1$.

<u>Geometry of $z^n = c \in \mathbb{C}$</u> = verticies of a regular n-gon with one vertex at $(r^{1/n}, \arg(c)/n)$ in polar co-ordinates.

So

$$\zeta_j = r^{1/n} e^{i\left(\frac{\arg(c)}{n} + 2\pi\frac{j}{n}\right)}$$

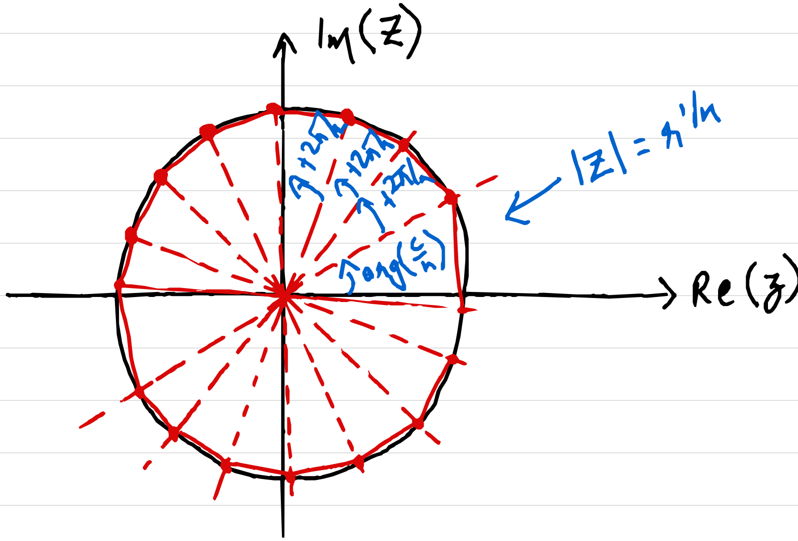
$\Leftrightarrow$

$$\zeta_j^{\,n} = r e^{i(\arg(c) + 2\pi j)} \qquad j \in \mathbb{Z}$$

using the fact of trig that $x = x + 2\pi k$ for $k \in \mathbb{Z}$

$\Leftrightarrow$

$$\zeta_j^{\,n} = z^n = re^{i\arg(c)} \Leftrightarrow z^n = c$$



Example: 4^{th} roots of i: $z^4 = i \in \mathbb{C}$

$$i = 1e^{i\pi/2} \quad \text{(in polar form)}$$

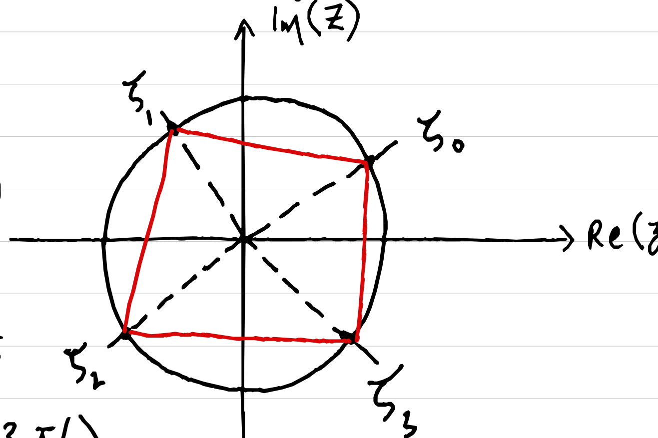
$$\zeta_j = 1^{1/4} e^{i(\pi/8 + 2\pi j/4)} \qquad j = 0, 1, 2, 3$$

$$\zeta_0 = e^{i\pi/8}$$

$$\zeta_1 = e^{i(\pi/8 + \pi/2)}$$

$$\zeta_2 = e^{i(\pi/8 + \pi}$$

$$\zeta_3 = e^{i(\pi/8 + 3\pi/2)}$$



(meant to be a square)

There is an alternative way of finding these roots for complex numbers:
By turning it into a system of equations in $\mathbb{R}$.

Say for example we want square root of $a+ib \in \mathbb{C}$ and $x+iy$ is one of them.

By defn of square root this means

$$a+ib = (x+iy)^2 = x^2-y^2 + i(2xy)$$

Comparing real and imaginary parts:

$$a = x^2-y^2 \qquad (*1)$$

$$b = 2xy \qquad (*2)$$

If $b \neq 0$, then $x \neq 0$, so we can have

$$y = b/2x$$

Substituting $y = b/2x$ in $(*1)$,

$$a = x^2 - \left(\frac{b}{2x}\right)^2 \text{ , and so}$$

$$x^4 - ax^2 - \frac{1}{4}b^2 = 0$$

This is a quadratic equation in x^2.
Set $\underline{t = x^2}$ equation becomes:

$$t^2 - at - \frac{1}{4}b = 0$$

Using quadratic formula:

$$\boxed{t = \frac{a \pm \sqrt{a^2 + b^2}}{2}}$$

t is square of a real number so $t > 0$, so choose the value of t such that $t > 0$.

let t_+ be the positive one. Then

$$\boxed{x = \pm\sqrt{t_+}}$$

$$y = \frac{b}{2x} \Rightarrow \boxed{y = \frac{\pm b}{2\sqrt{t_+}}}$$

$y = b/2x$, so therefore

square roots of $(a+ib)$ are

$$\sqrt{t_+} + \frac{ib}{2\sqrt{t_+}} \quad \text{and} \quad -\sqrt{t_+} - \frac{ib}{2\sqrt{t_+}}$$

Example Finding square roots of i using above method:

Let $a + ib = 0 + i.1$

Then $t = \dfrac{0 \pm \sqrt{0^2 + 1^2}}{2} = \pm\dfrac{1}{2}$

$t_+ = \dfrac{1}{2}$ and then square roots are

$$\sqrt{\frac{1}{2}} + \frac{i\,1}{2\sqrt{1/2}} \quad \text{and} \quad -\sqrt{\frac{1}{2}} - \frac{i\,1}{2\sqrt{\frac{1}{2}}}$$

$$\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \quad \text{and} \quad -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

<u>Example:</u> cube roots of $1+i$: $z^3 = 1+i$

$1+i$ in polar form:

$$\sqrt{2}\, e^{i\pi/4}.$$

So $$\zeta_j = 2^{1/6} e^{i\left(\pi/12 + \frac{2\pi j}{3}\right)} \qquad j = 0,1,2$$

$$\zeta_0 = 2^{1/6} e^{i\pi/12}$$

$$\zeta_1 = 2^{1/6} e^{i(\pi/12 + 2\pi/3)} = 2^{1/6} e^{i(9\pi/12)}$$

$$\zeta_2 = 2^{1/6} e^{i(\pi/12 + 4\pi/3)} = 2^{1/6} e^{i(17\pi/12)}$$
$$= 2^{1/6} e^{i(17\pi/12 - 2\pi)}$$
$$= 2^{1/6} e^{i(-7\pi/12)}$$

Curves and Planar Regions described using complex plane: (Loci)

Remember, $\mathbb{C}$ is just

$$\mathbb{C} = (\mathbb{R}^2, \times, +)$$

1) Circles:

$$(x-a)^2 + (y-b)^2 = R^2 \qquad (*1)$$

The $(x,y) \in \mathbb{R}^2$ which satisfy $(*_1)$ lie on a circle of radius R centred at (a,b)

If $z = x+iy$ then

$$(x-a)^2 + (y-b)^2 = |z-(a+ib)|^2 = R^2$$

$$\Rightarrow |z-(a+ib)| = R$$

Therefore

$\forall\ z = x+iy \in \mathbb{C}$

$$\boxed{|z-(a+ib)^2| = R}$$

are the set of points $(x,y) \in \mathbb{R}$ that form a circle of radius R centred at (a,b)

2) __Disc:__ __Closed discs__

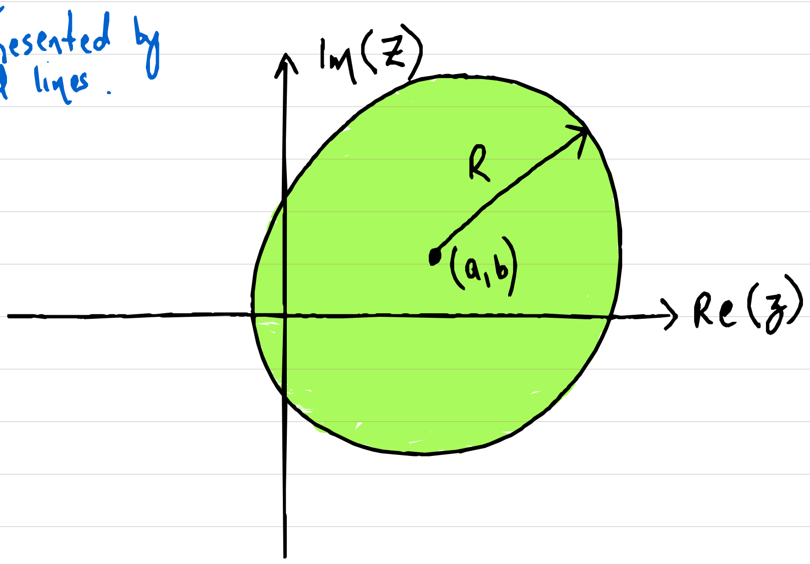
$$\boxed{|z-(a+ib)| \leq R}$$

all possible circles $\leq R$

$$\Rightarrow (x-a)^2 + (y-b)^2 \leq R^2 \quad (\star 2)$$

↳ basically set of all points $(x,y) \in \mathbb{R}$ lying inside of circle $(\star 2)$ including the boundary which is a circle.

represented by solid lines.



__Open disks:__

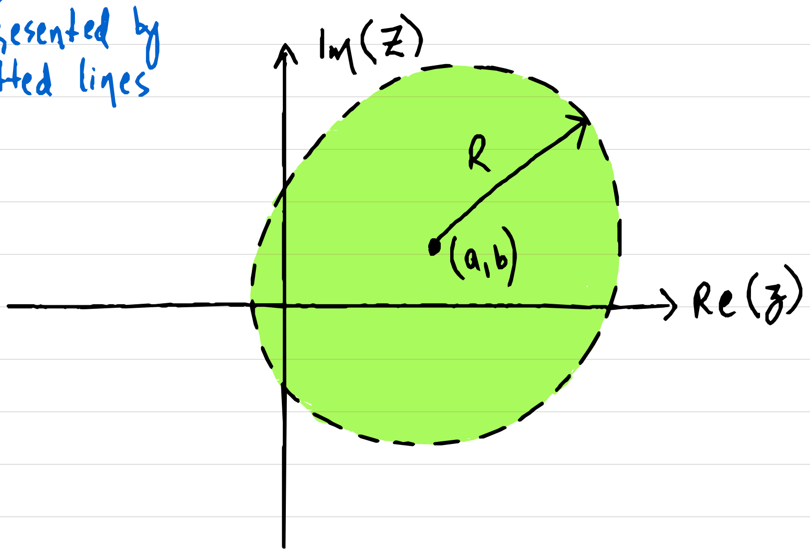
$$\boxed{|z-(a+ib)| < R}$$

all possible circles $\leq R$

$$\Rightarrow (x-a)^2 + (y-b)^2 < R^2 \quad (\star 2)$$

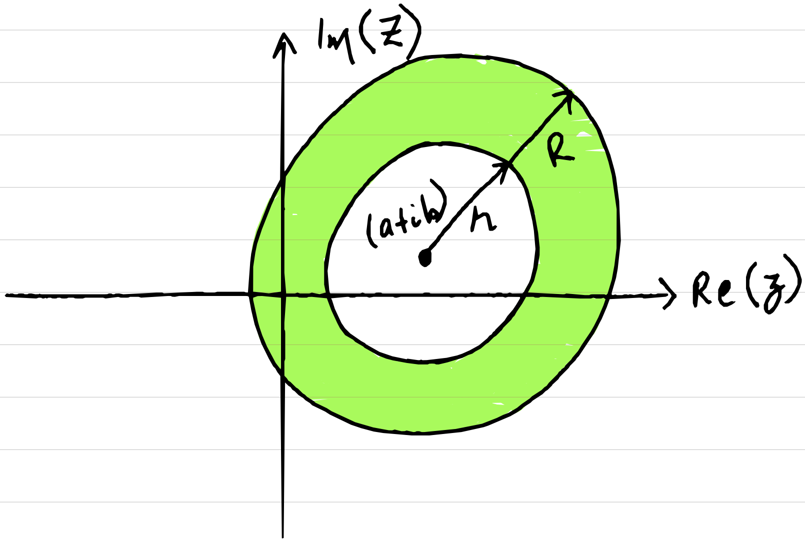
↳ basically set of all points $(x,y) \in \mathbb{R}$ lying inside of circle $(\star 2)$ NOT including the boundary which is a circle.

represented by dotted lines

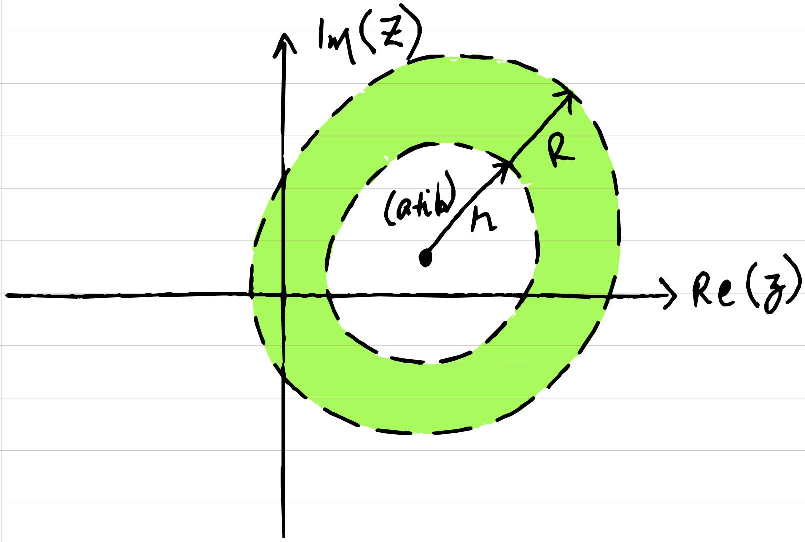


3) Annulus

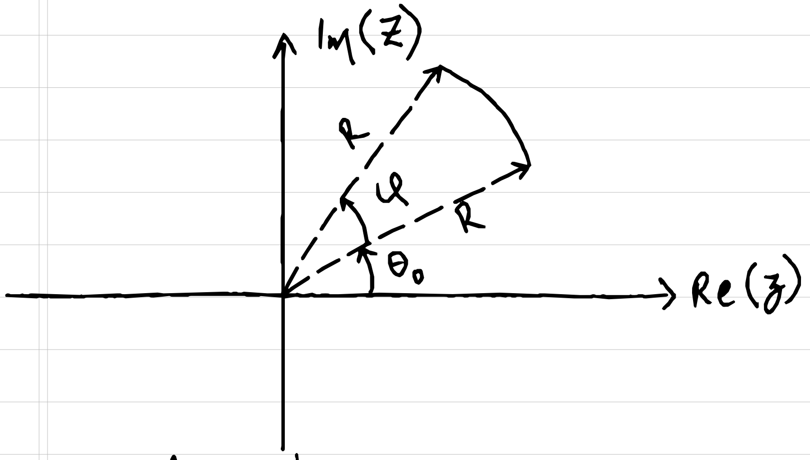
$$r \leq |z - (a+ib)| \leq R$$



$$r < |z - (a+ib)| < R$$



4) Arc:



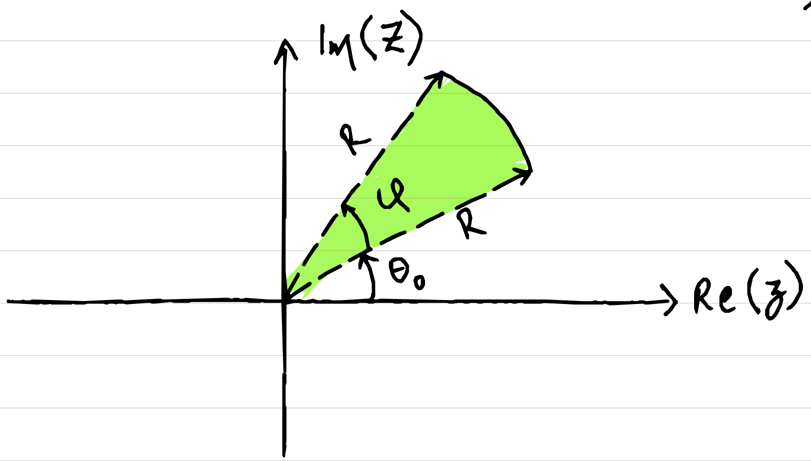
Set of points

$$\Gamma = \{ (R, \theta) : \theta_0 \leq \theta \leq \theta_0 + \varphi \}$$

$z = Re^{i\theta}$

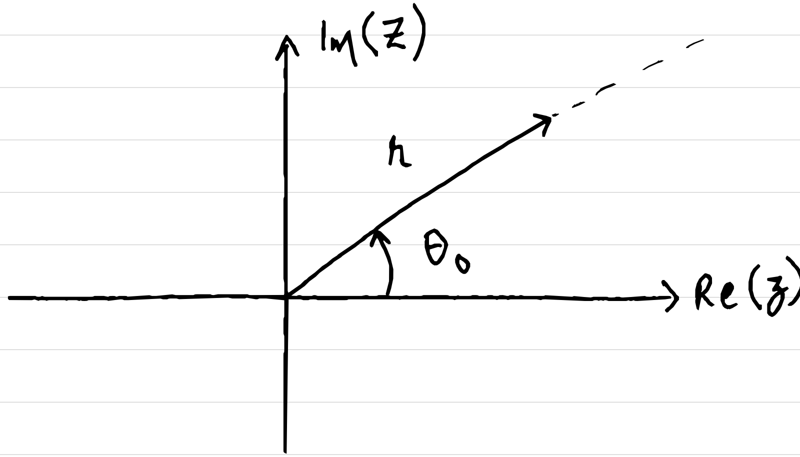
5) Wedge:

$$\Gamma' = \{ re^{i\theta} \mid 0 \leq r \leq R, \ \theta_0 \leq \theta \leq \theta_0 + \varphi \}$$



6) <u>Ray</u>:

$y = mx$ where $m \in \mathbb{R}$ and $x \in [0, \infty)$



$z = x + iy \implies \boxed{z = x + imx}$ where $x \in [0, \infty)$

$\implies z = re^{i\theta_0}$ where $r \in [0, \infty)$

General idea of finding loci:

Suppose

$$f(z) = c$$

where f is a function, $c \in \mathbb{C}$.

- Replace $z = x + iy$ or $z = re^{i\theta}$, try and determine the set of points z which satisfy

$$f(z) = c$$

Appendix of useful facts

$$i^n = \begin{cases} i & \text{if } n = 4k+1 \text{ for some } k \in \mathbb{Z} \\ -1 & \text{if } n = 4k+2 \text{ for some } k \in \mathbb{Z} \\ -i & \text{if } n = 4k+3 \text{ for some } k \in \mathbb{Z} \\ 1 & \text{if } n = 4k \text{ for some } k \in \mathbb{Z} \end{cases}$$